Exam <u>Numerical Mathematics 1</u>, July 5, 2016

Exercise 1

a ~~a~~ The error in polynomial interpolation of a function on x_k, $k=0,\dots,n$ is of the form

$q(x)\ g(x)$

where $q(x)$ is a polynomial and in general $g(x)$ ~~which~~ is a function

(i) Show that $q(x_k)\, g(x_k) = 0$

Answer: for the interpolation polynomial it holds that

$$p(x_k) = f(x_k)$$

So if ~~to~~

$$f(x) - p(x) = q(x)\, g(x)$$

then

$$0 = f(x_k) - p(x_k) = q(x_k)\, g(x_k) \quad \square$$

(ii) In general $g(x_k) \neq 0$ $k=0,\dots,n$,
Derive $q(x)$ from (i)

Answer: $q(x_k) = 0$ $k=0, \dots, n$

Is het duidelijk dat je de laagste graad moet hebben

The lowest degree polynomial satisfying this requirement is

$$(x-x_0)(x-x_1)\dots(x-x_n)$$

(iii) Give the part $g(x)$ of the error

Answer: $g(x) = \dfrac{f^{(n+1)}(\xi(x))}{(n+1)!}$

(iv) Suppose $x_k = -1 + kh$, $h = \frac{2}{n}$

Show that $\left| q(1 - \frac{1}{2}h) \right| \geq h^{n+1} \frac{1}{4}(n-1)!$

(v) Another choise of interpolation points are the zeros of a Chebyshev polynomial $T_{n+1}(x)$ which has that property that

$$T_n(\cos\varphi) = \cos(n\varphi)$$

and the coefficient of x^n in this polynomial is 2^{n-1}

Show that for this choise of ~~zero~~ interpolation points we have

$$| q(x) | \leq \frac{1}{2^n}$$

Not on exam

Answer (iv)

$$1 - \tfrac{1}{2}h = -1 + nh - \tfrac{1}{2}h = -1 + (n - \tfrac{1}{2})h$$

The common term in the error expansion

$$1 - \tfrac{1}{2}h - x_k = -1 + (n - \tfrac{1}{2})h - (-1 + kh) = (n - k - \tfrac{1}{2})h$$

$$\left| q(1 - \tfrac{1}{2}h - x_k) \right| = h^{n+1} \underbrace{(n - \tfrac{1}{2})}_{> n-1} \underbrace{(n - \tfrac{3}{2})}_{> n-2} \cdots \underbrace{\tfrac{3}{2}}_{> 1} \cdot \tfrac{1}{2} \cdot \left| -\tfrac{1}{2} \right|$$

$$\geq h^{n+1} \frac{1}{4} (n-1)!$$

Answer (v)

$$q(x) = \frac{1}{2^n} T_{n+1}(x) \rightarrow |q(x)| \leq \frac{1}{2^n} |T_{n+1}(x)| \leq \frac{1}{2^n}$$

b On the interval $[0, 3]$ a function is interpolated by on the point $\{0, 2, \quad\}$

i Give the interpolation polynomial

Answer:

One can write the interpolation polynomial as

$$\Pi_2(x) = f(0)\,\varphi_1(x) + f(2)\,\varphi_2(x) +$$

where $\varphi_i(x_k) = \delta_{ik}$

So $\hat{\varphi}_1(x) = (x-2)$

$$\varphi_1(x) = \frac{\hat{\varphi}_1(x)}{\hat{\varphi}_1(0)} = \frac{1}{-2}(x-2) = (2-x)/2$$

$$\left.\begin{array}{l}\hat{\varphi}_2(x) = x \\ \hat{\varphi}_2(2) = 2\end{array}\right\} \quad \varphi_2(x) = x/2$$

$$\to \Pi_2(x) = f(0)\frac{(2-x)}{2} + f(2)\frac{x}{2}$$

ii Derive the integration rule (for an integral of f based on the interpolation)

Answer: The integration rule follows from the exact integration of $\Pi_2(x)$ above

$$\int_0^3 f(0)\left(\frac{2-x}{2}\right) + f(2)\frac{x}{2}\,dx = f(0)\left(3 - \frac{x^2}{4}\Big|_0^3\right)$$

$$+ f(2)\frac{x^2}{4}\Big|_0^3 = \frac{3}{4}f(0) + \frac{9}{4}f(2)$$

iii What is the degree of exactness of this rule?

Answer

Since $H_2(x)$ is an arbitrary polynomial of degree 1 the exactness is at least 1

We can try whether it is also exact for quadratics.

On one side we have

$$\int_0^3 x^2 = \frac{1}{3}x^3 \Big|_0^3 = 9$$

The rule gives: $\frac{3}{4}0^3 + \frac{9}{4}2^3 = 18$

These are unequal

so degree of exactness is 1

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Exercise 2

Say A is a matrix of order 10 and it has the spectrum $\sigma(A) = \{1, 2, 3, 4, \ldots, 10\}$

a Consider the iteration

$x^{(0)}$ given

$y^{(n+1)}$ follows from solving $(A - 2.1 I)\, y^{(n+1)} = x^{(n)}$

$$x^{(n+1)} = y^{(n+1)} / \|y^{(n+1)}\|$$

(i) where will $x^{(n)}$ converge to?

(ii) what is the speed of convergence

(iii) Suppose instead of 2.1 we have 2.5. Show that $x^{(2n)}$ an $x^{(2n+1)}$ converge both but to a different vector

Answer (i) The matrix is of order 10 and there are 10 different eigenvalues, so there is a complete set of eigenvectors

Say $x^{(0)} = \sum_{i=1}^{10} \alpha_i v_i$ with $A v_i = \lambda_i v_i$ and $\|v_i\| = 1$

This is power method with the matrix $(A - 2.1 I)^{-1}$. The scaling is only a number so we can analyse the convergence without it.

$(A - 2.1 I)^{-1}$ has the same eigenv as A

$$(A - 2.1 I)^{-n} x^{(0)} = \sum_{i=1}^{10} \alpha_i (A - 2.1 I)^{-n} v_i = \sum_{i=1}^{10} \alpha_i (\lambda_i - 2.1)^{-n} v_i$$

$$(A - 2I)^{-n} v_i = (A - 2I)^{-(n-1)} (A - 2I)^{-1} v_i = \ldots = \frac{1}{\lambda_i - 2.1} v_i$$

$=$ here $\frac{1}{2-2.1} = -10$ will be the biggest eigenvalue

hence we split that one off

$$\alpha_2 (-10)^n v_2 + \sum_{\substack{i=1\\ i\neq 2}}^{10} \alpha_i \left(\frac{1}{\lambda_i - 2.1}\right)^n v_i =$$

$$= (-10)^n \left\{ \alpha_2 v_2 + \sum_{i\neq 2} \alpha_i \underbrace{\left(\frac{-0.1}{\lambda_i - 2.1}\right)}^n v_i \right\}$$

biggest one occurs for
$i = 3$; $\lambda_3 = 3$
$\frac{-0.1}{3-2.1} = -\frac{1}{9}$

So the sum will go to zero

$\Rightarrow (-10)^n \alpha_2 v_2$ for $n \to \infty$

Answer (i) The speed of convergence is determined by the largest ratio occurring in the remaining sum
$\longrightarrow \frac{1}{9}$

Answer (ii) If instead of 2.1 we have 2.5 then
$\frac{1}{2-2.5} = \frac{-1}{0.5}$ and $\frac{1}{3-2.5} = \frac{1}{0.5}$
are of equal magnitude

Hence we should look at

$$X^{(n)} = \alpha_2 \left(\frac{-1}{0.5}\right)^n v_2 + \alpha_3 \left(\frac{1}{0.5}\right)^n v_3 + \sum_{\substack{i=1\\ i\neq 2,3}}^{10} \cdots$$

$$= \left(\frac{1}{0.5}\right)^n \Bigg(\alpha_2 (-1)^n v_2 + \alpha_3 v_3 + \underbrace{\sum \alpha_i \left(\frac{0.5}{\lambda_i - 2.5}\right)^n v_i}_{\to 0} \Bigg)$$

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$$\Rightarrow \quad x^{(n)} \to \left(\frac{1}{0.5}\right)^n \left(\alpha_2 (-1)^n v_2 + \alpha_3 v_3\right)$$

hence $x^{(2n)} = \sim^{2n} \left(\alpha_2 v_2 + \alpha_3 v_3\right)$

$x^{(2n+1)} = \sim^{(2n+1)} \left(-\alpha_2 v_2 + \alpha_3 v_3\right)$

d A symmetric, can one apply Cholesky factorization

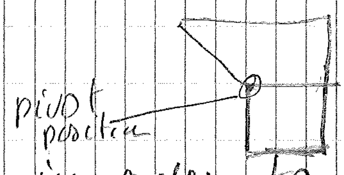
Answer Cholesky factorization $A = LL^T$
which only is possible if A is pos. def
Here the eigenvalues of $A - 2I$ are
$\{-1.1, -0.1, 0.9, \ldots\}$
and hence it is not pos. def. and
we cannot apply Cholesky factorization

e What is LU factorization with pivoting?

Answer: It holds that $LU = PA$
where $L = \Delta$ lower triangular
and $U = \nabla$ upper triangular

P is a permutation matrix which
is constructed along the way

Suppose we have as intermediate result



then in order to get a good pivot
we look for the max in the column
in abs. value

on and below the current pivot element.
By interchanging rows the maximum is brought to te pivot position.

Exercise 3

a. Consider the fixed point method
$x^{(n+1)} = \phi(x^{(n)})$ $x^{(0)}$ given.

Assume ϕ is one time continuous differentiable

a Show that $\left|\frac{d\phi}{dx}\right| < 1$ is needed for convergence in the neighbourhood of the fixed point

Answer let α be the fixed point : $\alpha = \phi(\alpha)$

$$\underbrace{x^{(n+1)} - \alpha}_{\equiv e^{(n+1)}} = \phi(x^{(n)}) - \phi(\alpha)$$
$$= \underbrace{\frac{d\phi}{dx}(\xi)\, e^{(n)}}_{\text{Taylor error}}$$

For convergence $|e^{(n+1)}| < |e^{(n)}|$
$$\rightarrow \left|\frac{d\phi}{dx}(x)\right| < 1 \text{ near the fixed point}$$

(b) Let $\phi(x) = x + a(x)\, f(x)$.

Which choice of $a(x)$ gives the fastest convergence near the fixed point.

Answer if $\frac{d\phi}{dx}(\alpha) = 0$

$$1 + \underbrace{a'(\alpha) f(\alpha)}_{=0} + a(\alpha) f'(\alpha) = 0$$
$$\rightarrow a(\alpha) = -\frac{1}{f'(\alpha)}$$

Choose $a(x) = -f(x)$

c) The fixed point iteration of the previous reads

$$x^{(n+1)} = x^{(n)} - \frac{f(x^{(n)})}{f'(x^{(n)})}$$

Show that

$$x^{(n+1)} = x^{(n)} - \frac{f(x^{(n)})\left(x^{(n)} - x^{(n-1)}\right)}{f(x^{(n)}) - f(x^{(n-1)})}$$

is a first order approximation of the former in terms of $(x^{(n)} - x^{(n-1)})$ if $f'(x) \neq 0$ in the neighbourhood of the fixed point

Notice error

Answer

$$f(x^{(n-1)}) = f(x^{(n)}) + f'(x^{(n)})(x^{(n-1)} - x^{(n)}) + O\left((x^{(n-1)} - x^{(n)})^2\right)$$

$$\Rightarrow \frac{f(x^{(n-1)}) - f(x^{(n)})}{x^{(n-1)} - x^{(n)}} = f'(x^{(n)}) + O(x^{(n-1)} - x^{(n)})$$

Hence $x^{(n+1)} = x^{(n)} - f(x^{(n)}) \dfrac{1}{f'(x^{(n)}) + O(\ )}$

$$= x^{(n)} - f(x^{(n)}) \frac{1}{f'(x^{(n)})(1 + O(\ ))} =$$

$$= x^{(n)} - \frac{f(x^{(n)})}{f'(x^{(n)})}(1 + O(\ )) = x^{(n)} - \frac{f(x^{(n)})}{f'(x^{(n)})} + O(\ )$$

d) One can write the previous method as

$$\begin{bmatrix} x^{(n+1)} \\ x^{(n)} \end{bmatrix} = \begin{bmatrix} x^{(n)} - f(x^{n})(\ ) \\ x^{(n)} \end{bmatrix} = \begin{bmatrix} \Phi_1(x^{(n)}, x^{(n-1)}) \\ \Phi_2(x^{(n)}, x^{(n-1)}) \end{bmatrix}$$

helaces

Define $e^{(n+1)} = \begin{bmatrix} x^{(n+1)} \\ x^{(n)} \end{bmatrix} - \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$

Derive the relation between $e^{(n+1)}$ and $e^{(n)}$ close to the fixed point

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or $z^{(n+1)} = \Phi(z^{(n)})$

where $z^{(n)} = \begin{bmatrix} x^{(n)} \\ x^{n-1} \end{bmatrix}$

and $\Phi(z) = \begin{bmatrix} z_1 - f(z_1)(z_1 - z_2)/(f(z_1) - f(z_2)) \\ z_1 \end{bmatrix}$

iii Show that $\lim_{\substack{z_1 \to \alpha \\ z_2 \to \alpha}} J_\phi(z) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

while taking the lim such that z_1 and z_2 remain different.
Hint (Exploit in a timely manner that $f(\alpha) = 0$)

Answer:

$$\frac{\partial \phi_1}{\partial z_1} = 1 - f'(z_1)(z_1 - z_2)/(f(z_1) - f(z_2)) - f(z_1)\frac{\partial}{\partial z_1}(z_1 - z_2 \ldots$$

NB $f(\alpha) = 0$ so this term cancels

$$\lim_{\substack{z_1 \to \alpha \\ z_2 \to \alpha}} \frac{\partial \phi_1}{\partial z_1}(z_1, z_2) = 1 - f'(\alpha)\frac{1}{f'(\alpha)} = 0$$

$$\frac{\partial \phi_1}{\partial z_2} = 0 - \underset{\to 0}{f(z_1)} \frac{\partial}{\partial z_2}(\quad)/(\quad)$$

$$\lim_{\substack{z_1 \to \alpha \\ z_2 \to \alpha}} \frac{\partial \phi_1}{\partial z_2}(z_1, z_2) = 0$$

$$\frac{\partial \phi_2}{\partial z_1} = 1$$

$$\frac{\partial \phi_2}{\partial z_2} = 0 \quad \rightarrow \quad J = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

not on exam

ii Will this method converge linearly ?

Answer: we have now that ($e^{(n)} = z^{(n)} - \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$)

$$e^{(n+1)} = J_\phi(\alpha, \alpha)\, e^{(n)} + h.o.t$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} e^{(n)} + h.o.t.$$

when $e^{(n)}$ small h.o.t. vanish we see convergence in 2 steps
this is superlinear

3C Vector case

Answer $\quad \vec{x}^{(n+1)} = \vec{\phi}(\vec{x}^{(n)})$

$\vec{\alpha} = \vec{\phi}(\vec{\alpha})$

$$\vec{x}^{(n+1)} - \vec{\alpha} = \vec{\phi}(\vec{x}^{(n)}) - \vec{\phi}(\vec{\alpha})$$

In general

① $$\phi_i(x_1 \ldots x_n) - \phi_i(\alpha_1 \ldots \alpha_n) = \sum_{j=1}^{n} \frac{\partial \phi_i(\vec{\alpha})}{\partial x_j}(x_j - \alpha_j) + h.o.t.$$

$$= \left[\frac{\partial \phi_i}{\partial x_1} \cdots \frac{\partial \phi_i}{\partial x_n}\right](\vec{x} - \vec{\alpha}) + \ldots$$

So $\quad \vec{x}^{(n+1)} - \vec{\alpha} = J_\phi(\vec{x}^{(n)} - \vec{\alpha}) + h.o.t.$

where $\quad (J_\phi)_{ij} = \frac{\partial \phi_i}{\partial x_j}(\vec{\alpha})$

If we neglect the h.o.t. then the above is just a linear iteration and we need for convergence of $x^{(n)} - \alpha$ to zero that $\rho(J_\phi) < 1$

② So all the eigenvalues of J_ϕ should be less than one in abs value.

Exercise 4

Consider the parabolic pde

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sin(\pi x t) \quad \text{for } 0<x<1 \text{ and } t>0$$

with initial condition $u(x,0) = \sin \pi x$

$$u(0,t) = u(1,t) = 0$$

(i) Consider the Trapezoidal method

$$w^{(n+1)} = w^{(n)} + \frac{\Delta t}{2}\left(f(w^{(n)}, t_n) + f(w^{(n+1)}, t_{n+1})\right)$$

Show that ~~this not~~ the truncation error $\tau(\Delta t)$ of this method is $O(\Delta t)^2$

ii Show that region of absolute stability of this method is precisely the left half plane in $\mathbb{C}$

Answer (i) exact solution of ODE $\frac{dy}{dt} = f(y,t)$

$$\tau(\Delta t) = \frac{y(t_{n+1}) - y(t_n)}{\Delta t} - \frac{1}{2}\left(f(y(t_n), t_n) + f(y(t_{n+1}), t_{n+1})\right)$$

$$= \frac{y(t_{n+1}) - y(t_n)}{\Delta t} - \frac{1}{2}\left(y'(t_n) + y'(t_{n+1})\right)$$

Taylor around $t_n + \frac{1}{2}\Delta t$. (this is the easiest but wrt another point leads to the same result)

In general we have

$$u(t \pm \tfrac{1}{2}\Delta t) = u(t) \pm \tfrac{1}{2}\Delta t\, u'(t) + \tfrac{1}{8}\Delta t^2 u''(t) \pm \tfrac{1}{48}\Delta t^3 u'''($$

Taking $t = t_n + \frac{1}{2}\Delta t$ and $u = y$ we find

$$y(t_{n+1}) - y(t_n) = \Delta t\, y'(t_n + \tfrac{1}{2}\Delta t) + \tfrac{1}{48}\Delta t^3\left(y'''(\xi^+) + y'''(\xi^-)\right)$$

Taking $u = y'$ we find

$$\frac{1}{2}\left(y'(t_n) + y'(t_{n+1})\right) = y'(t_n + \tfrac{1}{2}\Delta t) + \frac{1}{16}\Delta t^2\left(y'''(\eta^+) + y'''(\eta^-)\right)$$

We will find

$$\tau(\Delta t) = \Delta t^2\left(\frac{1}{48}\left(y'''(\xi^+) + y'''(\xi^-)\right) - \frac{1}{16}\left(y'''(\eta^+) + y'''(\eta^-)\right)\right)$$

$$= \Delta t^2 \, \frac{1}{12}\, y'''(t_n + \tfrac{1}{2}\Delta t) + O(\Delta t^3)$$

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Answer (ii)

Apply the method to the test equation $y' = \lambda y$

$$\Rightarrow w^{(n+1)} = w^{(n)} + \frac{1}{2}\Delta t \lambda \left(w^{(n)} + w^{(n+1)} \right)$$

$$\Rightarrow \left(1 - \frac{\Delta t}{2}\lambda\right) w^{(n+1)} = \left(1 + \frac{\Delta t}{2}\lambda\right) w^{(n)}$$

$$\Rightarrow w^{(n+1)} = \left(\frac{1 + \Delta t/2\,\lambda}{1 - \Delta t/2\,\lambda}\right) w^{(n)}$$

$$\Rightarrow r(z) = \frac{1 + z/2}{1 - z/2}$$

Region of abs stability $|r(z)| \leq 1$

$$\Rightarrow |1 + z/2| \leq |1 - z/2|$$

$$\Rightarrow |2 + z| \leq |2 - z|$$

distance of z to -2 distance of z to $+2$

distance is equal for $Re(z) = 0$

SO region of abs. stability is left half plane ie $Re(z) \leq 0$

iii Show that the space discretization on a grid with meshwidth h leads to a coupled system of ordinary differential equations

$$\frac{d}{dt} V = A V + F(t)$$

$$A = \frac{1}{h^2}\left(-2I + B\right)$$

Where $B = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{bmatrix}$

and give $F(t)$

T4

Answer

Using $u(x \pm h) = u(x) \pm h\,u'(x) + \frac{h^2}{2} u''(x) \pm \frac{h^3}{6} u'''(x) + \frac{h^4}{24} u^{IV}(\xi^{\pm})$

$\Rightarrow u(x+h) + u(x-h) = 2u(x) + h^2 u''(x) + \frac{h^4}{24}\left(u^{IV}(\xi^+) + u^{IV}(\xi^-)\right)$

$\rightarrow \frac{d^2u}{dx^2} = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$

Using this in the p.d.e. we obtain

$\frac{d}{dt} v_i = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} + \sin(\pi i h t)$ for $i = 1, \ldots, m$

Initial condition $v_i(0) = \sin \pi x_i = \sin \pi i h$

Boundary cond $v_0(t) = 0, \quad v_{m+1}(t) = 0$

In matrix form we obtain

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = \underbrace{\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{bmatrix}}_{-2I+B}\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} + \underbrace{\begin{bmatrix} \sin \pi h t \\ \sin \pi 2 h t \\ \vdots \\ \sin \pi m h t \end{bmatrix}}_{F(t)}$$

$$\begin{bmatrix} v_1(0) \\ \vdots \\ v_m(0) \end{bmatrix} = \begin{bmatrix} \sin \pi h \\ \vdots \\ \sin \pi m h \end{bmatrix}$$

iv Using that for general A ~~the eigenvalues~~ it holds that $\rho(A) \le \|A\|_\infty$ where $\|A\|_\infty = \max_i \sum_{j=1}^{m} |a_{ij}|$.

The eigenvalues of the specific A are in the interval $\left[-\frac{4}{h^2}, 0\right]$

v Answer $\rho(B) \le \|B\|_\infty = 2 \rightarrow -2 \le \lambda(B) \le 2$

Furthermore $\lambda(A) = \frac{1}{h^2}\left(-2I + \lambda(B)\right)$

Combining $\lambda(A) \in \frac{1}{h^2}[-4, 0] = \left[-\frac{4}{h^2}, 0\right]$

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V Show that Trapezoidal method applied to the ODE does not have a time step restriction

Answer:

we have that $\Delta t \lambda(A) \leq 0$ for all eigenvalues of A

so $\Delta t \lambda(A)$ is in the region of abs. stab. for all Δt and hence there is no time step restriction.